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Growth of Discontinuities in Relativistic Fluids

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The paper modifies the work on discontinuities in relativistic fluids [*International Journal of Theoretical Physics*, Vol. 19(8), p. 563 (1980)], and puts its analysis and conclusions on a right footing.

1. INTRODUCTION

In a recent paper, Ram and Singh (1980) have extended the work of McCarthy (1969) to a relaxing gas model of Chu (1970, pp. 35–46). Although Ram et al. have used the arguments of McCarthy in deriving their growth equation, but they have failed to properly translate these arguments to their problem, and thus their several intermediate steps and final conclusions are in error. For instance: (i) The set of their basic equations essentially needs an additional equation $Ds=(\xi/T)(\beta w/c)$ for the entropy s [see Chu (1980), p. 37]; (ii) the geometrical compatibility conditions (3.1) and (3.2) are incorrectly stated in the sense that there is no distinction between the indices used for space tensor $g_{\alpha\beta}$, and surface tensor $a_{\alpha\beta}$, where the range of Greek indices is 1,2,3,4 (see Section 2 of Ram et al.); it ought to be well known that the surface tensors in four-dimensional Einstein–Riemann space have nine components; thus they should be denoted by $a_{\Gamma\Lambda}$ and $b^{\Gamma\Lambda}$, where capital Greek indices take values 1,2,3; surprisingly in their paper, the second fundamental form of the surface has been represented by $b_{\Gamma\tau}$ (in Section 3) and $b_{\phi\tau}$ (in Section 4), indicating 12 and 16 components, respectively; (iii) the expression for $\delta[Z]$ in equation (3.3) is incorrect; (iv) equations (4.1)–(4.3) are incorrect, and they need an essential modification; (v) the solution (5.1) of (4.5) is incorrectly stated, and therefore equations (5.2) to (5.7) need essential corrections; (vi) in Section 5, the sign of the

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constant term A_3 is assumed to be positive without any apparent reason; (viii) the statement "when the condition (5.3) is not satisfied, the wave amplitude will decrease and the wave will be damped out ultimately" appearing in Section 5 is obviously incorrect, because equality in (5.3) corresponds to a situation, where the wave can neither terminate into a shock nor can it ever damp out.

The purpose of the present paper is to rectify these shortcomings in the work of Ram et al., mainly because they are likely to lead to misconceptions in the minds of readers and future workers of the field.

2. BEHAVIOR AT THE WAVE FRONT

If $[Z]$ denotes the jump in Z across a weak discontinuity surface S, defined as $x^\mu = \psi^\mu(b^1, b^2, b^3)$ with $\mu = 1,2,3,4$, then the geometrical conditions of compatibility derived by Truesdell and Toupin (1960, p. 497) reduce to

$$[Z,_\alpha] = BN_\alpha$$

$$\left[Z,_{\alpha\beta}\right] = \bar{B}N_\alpha N_\beta + 2N_{(\alpha}x^\Gamma_{\beta)}B,_\Gamma - Bb_{\Gamma\Sigma}x^\Gamma_{(\alpha}x^\Sigma_{\beta)}$$

where $B = [Z,_\alpha]N^\alpha$, $\bar{B} = [Z,_{\alpha\beta}]N^\alpha N^\beta$, $2M_{(\alpha\beta)} = M_{\alpha\beta} + M_{\beta\alpha}$, $x^\Gamma_\beta = g_{\alpha\beta}a^{\Gamma\Sigma}x^\alpha_{;\Sigma}$, $a_{\Gamma\Sigma} = g_{\alpha\beta}x^\alpha_{;\Gamma}x^\beta_{;\Sigma}$, and $b_{\Gamma\Sigma} = -g_{\alpha\beta}x^\alpha_{;\Gamma}N^\beta_{;\Sigma}$. It may be noted that capital Greek indices Γ, Σ, etc., which refer to surface coordinates b^1, b^2, b^3, have the range $1,2,3$, while small Greek indices α, β, etc., which refer to space-time coordinates x^μ, have the range $1,2,3,4$.

Attention is now given to the analysis of Ram et al. (1980). We feel that the following errors in their paper need to be corrected.

(i) In equation (3.3), the expression for $\delta[Z]$ should be $U^\mu[Z],_\mu - VN^\mu[Z],_\mu$ instead of $U^\mu[Z,_\mu] - VN^\mu[Z,_\mu]$; in fact, the identity used in deriving (3.3) should read as $a^{\Gamma\Sigma}x^\alpha_{;\Gamma}x^\beta_{;\Sigma} = g^{\alpha\beta} - N^\alpha N^\beta$.

(ii) Equations (4.1) and (4.2) are in error; the corrected forms should be

$$\rho\sigma c^2 V\bar{\lambda}^\alpha N_\alpha + (1+V^2)\bar{\mu} + V\delta(\mu) + \rho\sigma c^2\delta(\lambda) - \rho\sigma c^2\lambda^\alpha\delta(N_\alpha) - 3\rho a_f^2\lambda^2 = 0$$

$$V\bar{\mu} + \delta(\mu) + (1+\Gamma)\lambda\mu + \rho a_f^2(1+V^2)^{-1}\left(\lambda\overset{*}{N}{}^\alpha_{;\Gamma}x^\Gamma_\alpha + x^\Gamma_\alpha\overset{*}{N}{}^\alpha\lambda_{;\Gamma}\right)$$

$$+ \rho a_f^2\bar{\lambda}^\alpha N_\alpha - \frac{\beta}{c}\frac{\rho a_f^4}{V}\frac{\partial\rho}{\partial q}\frac{\partial w}{\partial p}\lambda = 0$$

(iii) Equation (4.3), which they call fundamental differential equation governing the growth and decay of a weak discontinuity, is in error; the corrected form should be

$$\left(2\sigma - \frac{a_f^2}{c^2}\right)\delta(\lambda) + \lambda\left[(\sigma\beta/c)\Lambda - a_f^2(Vc^2)^{-1}\overset{*}{N}{}^{\alpha}_{;\Gamma}x^{\Gamma}_{\alpha} - \sigma_0(1+V^2)^{-1}\overset{*}{N}{}^{\alpha}\delta(N_\alpha)\right]$$

$$-a_f^2(Vc^2)^{-1}x^{\Gamma}_{\alpha}\overset{*}{N}{}^{\alpha}\lambda_{;\Gamma} + \lambda^2\left[\sigma_0(1+\Gamma_0) - 3\left(a_f^2/c^2\right)\right] = 0$$

where $\Lambda = a_f^2(\partial\rho/\partial q)(\partial w/\partial p)$. According to Chu (1970), $\Lambda = \tau^{-1}[(a_f^2/a_e^2) - 1] > 0$, where a_e and τ are, respectively, the equilibrium sound speed and the relaxation time.

It may be noted that the relations $\overset{*}{N}{}^{\alpha}_{;\Gamma}x^{\Gamma}_{\alpha} = -2\Omega$ and $\overset{*}{N}{}^{\alpha}\delta(N_\alpha) = 0$ are true only in a local instantaneous rest frame, where $\overset{*}{N}{}^{\alpha} = (1+V^2)(n^i, 0)$. But Ram et al. have used these relations in their equations (4.1) and (4.2), before they make any such assumption, and retain the term $\overset{*}{N}{}^{\alpha}x^{\Gamma}_{\alpha}\lambda_{;\Gamma}$; it may also be noted that in a local instantaneous $\overset{*}{N}{}^{\alpha}x^{\Gamma}_{\alpha}\lambda_{;\Gamma}$ vanishes.

(iv) In equations (5.1) to (5.4), the constant A_3 should be replaced by A_3G_0 and in the expression of $\phi(t)$, the power of the quanity within brace brackets should be $-1/(2A_1)$ instead of $-A_1/2$.

(v) The discussion of (5.1) should run as follows:

Since S is a timelike hypersurface, its speed of propagation is less than the speed of light, i.e., $G_0 < c$; hence the coefficients A_1 and A_2 are positive, because $a_f > a_e$. Thus, for converging waves, the integral

$$I(t^*) \equiv \int_0^{t^*} \phi(t)\,dt = \int_0^{t^*} [(1 - K_1G_0t)(1 - K_2G_0t)]^{-1/(2A_1)}$$
$$\times \exp(-A_2G_0t/A_1)\,dt$$

converges when $A_1 > 1/2$, and it diverges when $A_1 \leqslant 1/2$. Hence (a) when $A_1 > 1/2$ and $\operatorname{sgn} b(0) = -\operatorname{sgn} A_3$, there exists a critical value b_c of the initial amplitude given by $b_c = [(|A_3|G_0/A_1)I(t^*)]^{-1}$, such that waves with initial amplitude less than b_c form a focus but not a shock as $t \to t^*$; waves with initial amplitude equal to b_c form a shock and a focus simultaneously as $t \to t^*$; and waves with initial amplitude greater than b_c form a shock before the focus at $t = t_c < t^*$, where t_c is given by $I(t_c) = A_1(|A_3b(0)|G_0)^{-1}$; (b) when $A_1 \leqslant 1/2$ and $\operatorname{sgn} b(0) = -\operatorname{sgn} A_3$, it is interesting to note that the critical amplitude vanishes; this means that all waves no matter how small be their initial amplitude, terminate in a shock before the formation of the focus.

For diverging waves, when $\operatorname{sgn} b(0) = -\operatorname{sgn} A_3$, there exists a critical value b_c of the initial amplitude given by $b_c = [(|A_3|G_0/A_1)I(\infty)]^{-1}$ such that for $|b(0)| < b_c$, the wave decays [i.e., $b(t) \to 0$ as $t \to \infty$]; for $|b(0)| = b_c$, the wave ultimately takes a stable wave form [i.e., $b(t) \to A_2/(A_3G_0)$ as $t \to \infty$]; and for $|b(0)| > b_c$, the wave terminates in a shock in a finite time t_c (i.e., $|b| \to \infty$ as $t \to t_c$), where t_c is given by $I(t_c) = A_1[|A_3b(0)|G_0]^{-1}$.

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